Even sharper upper bounds on the number of points on curves

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Symposium on Algebraic Geometry and its Applications
Tahiti, May 2007
Revised slides

How many points can there be on a genus-g curve?

For a prime power q and an integer $g \geq 0$, set

$$N_q(g) = \max\{\#C(\mathbb{F}_q) : C \text{ is a genus-}g \text{ curve over } \mathbb{F}_q\}.$$

Questions

What can we say about $N_q(g) \ldots$

- Asymptotically?
- For specific values of q and g?

Asymptotic results (q fixed, $g \to \infty$).

We set $A(q) = \limsup_{g\to\infty} N_q(g)/g$.

Weil

We have $N_q(g) \le q + 1 + 2g\sqrt{q}$, so $A(q) \le 2\sqrt{q}$.

Serre

We have $N_q(g) \le q + 1 + g\lfloor 2\sqrt{q}\rfloor$, so $A(q) \le \lfloor 2\sqrt{q}\rfloor$.

Ihara

We have $A(q) \le (\sqrt{8q+1} - 1)/2$.

Drinfel'd-Vlăduţ

We have $A(q) \le \sqrt{q} - 1$, with equality when q is square.

Specific values of q and g.

Goal: Find upper and lower bounds on $N_q(g)$.

Lower bounds

Clever people construct curves with many points, using...

- Class field theory
- Towers of curves
- Fiber products of Artin-Schreier curves
- Modular curves
- Other explicit curves
- ...

Many, many people have contributed to the best known lower bounds for various q and g.

Specific values of q and g.

Upper bounds

- Weil-Serre bound
- Oesterlé bound
- Other restrictions (Stöhr-Voloch, Fuhrmann-Torres, Korchmáros-Torres, . . .)

Are these upper bounds on $N_q(g)$ the best possible?

Or can we sometimes do better?

Weil polynomials.

The Weil polynomial of an abelian variety A over $\mathbb{F}_q$ is the characteristic polynomial of its Frobenius endomorphism.

The Weil polynomial of curve over $\mathbb{F}_q$ is the Weil polynomial of its Jacobian.

If A has dimension n, then its Weil polynomial has the form

$$x^{2n} + a_1 x^{2n-1} + \cdots + a_{n-1} x^{n+1} + a_n x^n + a_{n-1} q x^{n-1} + \cdots + a_1 q^{2n-1} x + q^{2n}.$$

All of its roots in $\mathbb{C}$ lie on the circle $|z| = \sqrt{q}$. Its real roots have even multiplicity.

Note: The Honda-Tate theorem provides further restrictions.

More on Weil polynomials.

Since the roots of f come in complex-conjugate pairs, we may write

$$f(x) = x^n h(x + q/x)$$

for a unique monic $h \in \mathbb{Z}[x]$, the real Weil polynomial of C. The roots of h are real numbers in the interval $[-2\sqrt{q}, 2\sqrt{q}]$.

Note that if $f = x^{2n} + a_1 x^{2n-1} + \cdots$, then $h = x^n + a_1 x^{n-1} + \cdots$.

Theorem (Tate)

Two abelian varieties over $\mathbb{F}_q$ are isogenous to one another if and only if they have the same Weil polynomial.

Weil polynomials of curves.

Suppose C is a genus-g curve over $\mathbb{F}_q$, with Weil polynomial f. Write $f = \prod_{i=1}^{2g}(x - \pi_i)$ with $\pi_i \in \mathbb{C}$. Then for all $d > 0$ we have

$$\#C(\mathbb{F}_{q^d}) = q^d + 1 - \sum \pi_i^d.$$

In particular, if $f = x^{2g} + a_1 x^{2g-1} + \cdots$, then

$$\#C(\mathbb{F}_q) = q + 1 + a_1.$$

These formulas can be used to compute the number of degree-d places on the curve, for each d.

Serre's strategy for bounding $N_q(g)$.

Goal: Show that no genus-g curve over $\mathbb{F}_q$ has exactly N points.

- Compute all $h = x^g + a_1 x^{g-1} + \cdots$ with all complex roots in the real interval $[-2\sqrt{q}, 2\sqrt{q}]$, where $a_1 = N - q - 1$.
- Find a reason why each h can't come from a curve.
  - The Honda-Tate conditions.
  - The number of degree-d places on a curve must be ≥ 0.
  - The "resultant 1" method.
    - Eliminate h if $h = h_1 h_2$ with $\mathrm{Res}(h_1, h_2) = 1$.
  - Restrictions when h is the real Weil polynomial of E^g.
  - Miscellaneous *ad hoc* methods.

Extensions to Serre's techniques.

In 2003, Kristin Lauter and I added some further methods:

- The "resultant 2" method.
  - If $h = h_1 h_2$ and $\mathrm{Res}(\sqrt{h_1}, \sqrt{h_2}) = 2$, then C must be a double cover of a curve with real Weil polynomial h_1 or h_2.
  - (Here $\sqrt{h_i}$ denotes the *radical* of h_i.)
- The "elliptic factor" method.
  - If $h = (x - t)h_2$ for the real Weil polynomial $x - t$ of an elliptic curve E, and if $r = \mathrm{Res}(x - t, \sqrt{h_2})$, then C has a map of degree dividing r to an elliptic curve isogenous to E.

Sometimes, contradictions follow.

Example.

Consider $q = 8$, $g = 9$, $N = 46$.

Let $h = (x + 3)^4(x + 5)^5$. All of its roots lie in $[-2\sqrt{8}, 2\sqrt{8}]$. Why isn't it the real Weil polynomial of a genus-9 curve C over $\mathbb{F}_8$?

Answer: The resultant 2 method.

Such a C would be a double cover of a curve with real Weil polynomial either $(x + 3)^4$ or $(x + 5)^5$.

A curve with real Weil polynomial $(x + 5)^5$ would have fewer points over $\mathbb{F}_{64}$ than over $\mathbb{F}_8$, so $(x + 5)^5$ fails.

A curve with real Weil polynomial $(x + 3)^4$ has 21 points. A curve with 46 points can't be a double cover of a curve with 21 points.

The van der Geer/van der Vlugt tables

Upper and lower bounds on $N_q(g)$, as of January 2002.

$g \setminus q$	2	4	8	16	32	64	128
1	5	9	14	25	44	81	150
2	6	10	18	33	53	97	172
3	7	14	24	38	64	113	192
4	8	15	25	45 – 46	71 – 75	129	215 – 217
5	9	17 – 18	29 – 32	49 – 54	83 – 86	132 – 145	227 – 239
6	10	20	33 – 35	65	86 – 97	161	243 – 261
7	10	21 – 22	34 – 39	63 – 70	98 – 108	177	258 – 283
8	11	21 – 24	34 – 43	61 – 76	97 – 119	169 – 193	257 – 305
9	12	26	45 – 47	72 – 81	108 – 130	209	288 – 327
10	13	27 – 28	42 – 50	81 – 87	– 141	225	289 – 349
11	14	26 – 30	48 – 54	80 – 92	120 – 152	201 – 239	– 371
12	14 – 15	29 – 31	49 – 57	83 – 97	129 – 163	257	321 – 393
13	15	33	56 – 61	97 – 103	129 – 174	255 – 270	– 415
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The van der Geer/van der Vlugt tables

Upper bounds from 2002, lower bounds from November 2006.

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New methods.

Lauter and I have been revisiting this topic.

New methods

- The “reduced resultant 2” method.
- The “generalized elliptic factor” method.

Rest of the talk:

- Explain the new (and old) methods.
- Show some new results.

The basic idea.

Question underlying the old and new methods:

How close is Jac C to a product of polarized varieties?

Suppose h is the real Weil polynomial of an isogeny class $\mathcal{I}$.

If $h = h_1 h_2$ for two coprime factors, then $\mathcal{I}$ contains $A_1 \times A_2$, where $\mathrm{Hom}(A_1, A_2) = 0$.

(The real Weil polynomial for A_i is h_i.)

Finding the smallest kernel.

$$0 \longrightarrow \Delta' \longrightarrow A_1 \times A_2 \longrightarrow \operatorname{Jac} C \longrightarrow 0$$

Finding the smallest kernel.

$$
\begin{array}{ccccccccc}
 & & \Delta_1 \times \Delta_2 & = & \Delta_1 \times \Delta_2 & & & & \\
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0 & \longrightarrow & \Delta & \longrightarrow & B_1 \times B_2 & \longrightarrow & \mathrm{Jac}\, C & \longrightarrow & 0
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\end{array}
$$

Each B_i is the image of A_i in Jac C.

Projections $B_1 \times B_2 \to B_i$ give *injections* $\Delta \hookrightarrow B_1$ and $\Delta \hookrightarrow B_2$.

Goal: Understand Δ.

Bounding the size of the kernel Δ.

Let π, π_1, π_2 be Frobenius on Jac C, B_1, B_2, respectively.

$$
\begin{array}{ccc}
\operatorname{End}\operatorname{Jac} C & \hookrightarrow & (\operatorname{End} B_1) \times (\operatorname{End} B_2) \\
\uparrow & & \uparrow \\
\mathbb{Z}[\pi, \overline{\pi}] & \hookrightarrow & \mathbb{Z}[\pi_1, \overline{\pi}_1] \times \mathbb{Z}[\pi_2, \overline{\pi}_2] \\
\pi & \longmapsto & (\pi_1, \pi_2)
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Find φ such that $\varphi \longmapsto (0, n)$ for some n.

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Find φ such that $\varphi \longmapsto (0, n)$ for some n.

Then φ acts as 0 on $B_1 \hookleftarrow \Delta$,
and φ acts as n on $B_2 \hookleftarrow \Delta$,
so Δ is killed by n.

A simpler computation.

$$\begin{array}{ccc}
\mathbb{Z}[\pi,\overline{\pi}] & \hookrightarrow & \mathbb{Z}[\pi_1,\overline{\pi}_1]\times\mathbb{Z}[\pi_2,\overline{\pi}_2] \\
\uparrow & & \uparrow \\
\mathbb{Z}[\pi+\overline{\pi}] & \hookrightarrow & \mathbb{Z}[\pi_1+\overline{\pi}_1]\times\mathbb{Z}[\pi_2+\overline{\pi}_2]
\end{array}$$

Find $n > 0$ for which there is a $\varphi \in \mathbb{Z}[\pi + \overline{\pi}]$ that maps to $(0, n)$.

Let $m_i = (\text{minimal polynomial of } \pi_i + \overline{\pi}_i) = \sqrt{h_i}$.

$$\mathbb{Z}[x]/(m_1 m_2) \hookrightarrow \mathbb{Z}[x]/(m_1) \times \mathbb{Z}[x]/(m_2)$$

Smallest n is the generator of the ideal $\mathbb{Z} \cap (m_1, m_2)$.

Reduced resultants.

Definition

The *reduced resultant* $\text{Res}'(f, g)$ of two polynomials $f, g \in \mathbb{Z}[x]$ is the non-negative generator of the ideal $\mathbb{Z} \cap (f, g)$.

To compute $\text{Res}'(f, g)$:
Write $af + bg = 1$ in $\mathbb{Q}[x]$, and then clear denominators.

The reduced resultant divides the usual resultant, and is divisible by the radical of the usual resultant.

Note

The n we get from $\mathbb{Z}[\pi, \overline{\pi}]$ is either $\text{Res}'(m_1, m_2)$ or half this, and we can easily tell which.

The n we get from $\mathbb{Z}[\pi, \overline{\pi}]$ is the modified reduced resultant.

New versions of old results.

Let $h = h_1 h_2$ be the real Weil polynomial of an isogeny class $\mathcal{I}$, where h_1 and h_2 are coprime.

Let r be the modified reduced resultant of $\sqrt{h_1}$ and $\sqrt{h_2}$.

Theorem (Serre)

If $r = 1$ then there is no Jacobian in $\mathcal{I}$.

Theorem

If $r = 2$ and if $\operatorname{Jac} C$ lies in $\mathcal{I}$, then C is a double cover of a curve D whose real Weil polynomial is either h_1 or h_2.

Proof.

Consider the principal polarization λ on Jac C.

$$\text{Jac } C \xrightarrow[\sim]{\lambda} \widehat{\text{Jac } C}$$

Proof.

Consider the principal polarization λ on Jac C.

$$\begin{array}{ccc}
B_1 \times B_2 & \xrightarrow{\mu_1 \times \mu_2} & \widehat{B_1} \times \widehat{B_2} \\
\downarrow & & \uparrow \\
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If $r = 1 \ldots$

Then $(\operatorname{Jac} C, \lambda) \cong (B_1 \times B_2, \mu_1 \times \mu_2)$, impossible. □

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If $r = 1 \ldots$

Then $(\mathrm{Jac}\, C, \lambda) \cong (B_1 \times B_2, \mu_1 \times \mu_2)$, impossible. □

If $r = 2 \ldots$

Consider the involution $(1, -1)$ of $(B_1 \times B_2, \mu_1 \times \mu_2)$:

- acts trivially on Δ;
- gives an involution of $(\mathrm{Jac}\, C, \lambda)$;
- gives an involution of C, and so a double cover $C \to D$. □

The generalized elliptic factor method.

Suppose $\mathcal{I}$ contains $E^n \times A$, where $\mathrm{Hom}(E, A) = 0$.

Gives $h = h_1 h_2$ with $h_1 = (x - t)^n$, where $t = \mathrm{trace}(E)$.

Let r be the modified reduced resultant of $\sqrt{h_1}$ and $\sqrt{h_2}$.

Theorem

Suppose Jac C *lies in* $\mathcal{I}$.

- *If* $n = 1$, *then there is a map from* C *to an elliptic curve isogenous to* E, *of degree dividing* r.
- *If* $n > 1$, *then there is a map from* C *to an elliptic curve isogenous to* E, *whose degree can be effectively bounded.*

Sketch of proof.

Recall that in general we had

$$\begin{array}{ccc} B_1 \times B_2 & \xrightarrow{\mu_1 \times \mu_2} & \widehat{B_1} \times \widehat{B_2} \\ \downarrow & & \uparrow \\ \operatorname{Jac} C & \underset{\sim}{\xrightarrow{\lambda}} & \widehat{\operatorname{Jac} C} \end{array}$$

where the kernel Δ of $B_1 \times B_2 \to \operatorname{Jac} C$ injects into B_1 and B_2.

Then $\Delta \hookrightarrow \ker \mu_1$ and $\Delta \hookrightarrow \ker \mu_2$ as well.
Counting degrees, we find that $\ker \mu_1 \cong \Delta \cong \ker \mu_2$.

In present case $B_1 \sim E^n$.

Sketch of proof, $n = 1$.

$$\begin{array}{ccc} F \times B_2 & \xrightarrow{\mu_1 \times \mu_2} & \widehat{F} \times \widehat{B_2} \\ \downarrow & & \uparrow \\ \operatorname{Jac} C & \xrightarrow[\sim]{\lambda} & \widehat{\operatorname{Jac} C} \end{array}$$

Sketch of proof, $n = 1$.

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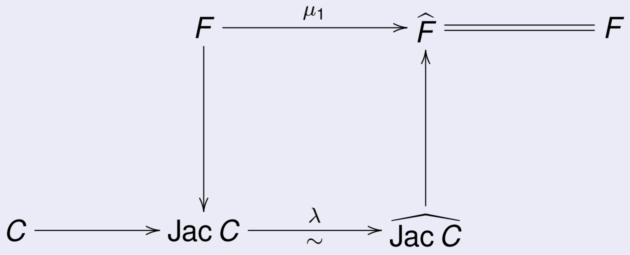

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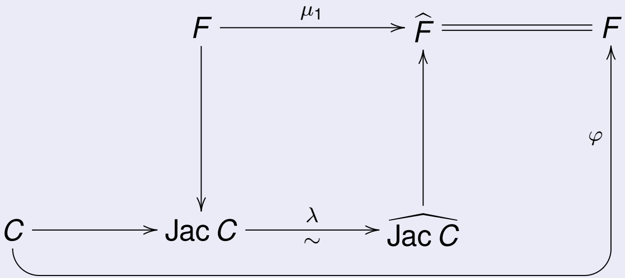

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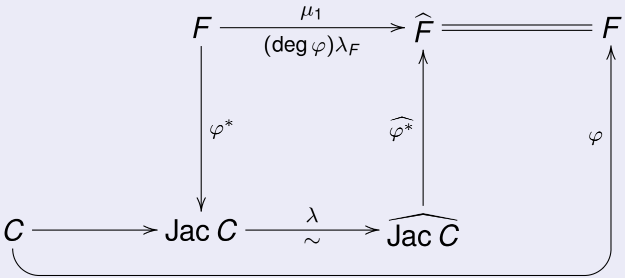

μ_1 is multiplication-by-deg φ, followed by canonical polarization.

Since kernel μ_1 is killed by r, deg φ divides r.

Sketch of proof, $n > 1$.

Recall the statement we want to prove:

We have:

- $h = h_1 h_2$ with $h_1 = (x - t)^n$, where $t = \text{trace}(E)$.
- $n > 1$.
- r is the modified reduced resultant of $(x - t)$ and $\sqrt{h_2}$.
- A curve C has real Weil polynomial $h_1 h_2$.

We want to show:

- There is a map from C to an elliptic curve isogenous to E, whose degree can be effectively bounded.

Sketch of proof, $n > 1$.

Let us consider the case where $t^2 - 4q$ is a fundamental discriminant, corresponding to a quadratic order $\mathcal{O}$ of class number 1.

Then $B_1 \cong E^n$, and a polarization μ_1 on B_1 can be viewed as a positive definite Hermitian form H on $\mathcal{O}^n$.

We have $\deg \mu_1 = (\det \operatorname{Gram} H)^2$.

Suppose $\gamma = (a_1, a_2, \ldots, a_n) \in \mathcal{O}^n$ has squared-length m under H.

Consider the map $\Gamma : E \to E^n$ given by γ. Then the pullback of μ_1 by Γ is m times the canonical polarization of E.

The big diagram when $n > 1$.

$$\begin{array}{ccc}
E^n \times B_2 & \xrightarrow{\mu_1 \times \mu_2} & \widehat{E}^n \times \widehat{B_2} \\
\downarrow & & \uparrow \\
\operatorname{Jac} C & \xrightarrow[\sim]{\lambda} & \widehat{\operatorname{Jac} C}
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The big diagram when $n > 1$.

$$\begin{array}{ccc} E & \xrightarrow{\text{degree } m^2} & \widehat{E} \\ \Big\downarrow{\scriptstyle \Gamma} & & \Big\uparrow{\scriptstyle \widehat{\Gamma}} \\ E^n & \xrightarrow{\mu_1} & \widehat{E}^n \\ \Big\downarrow & & \Big\uparrow \\ \operatorname{Jac} C & \xrightarrow[\sim]{\lambda} & \widehat{\operatorname{Jac} C} \end{array}$$

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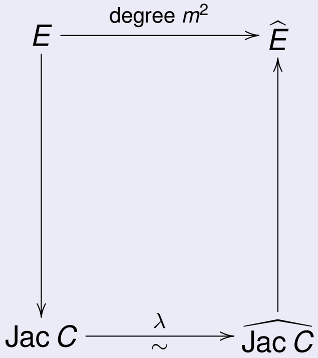

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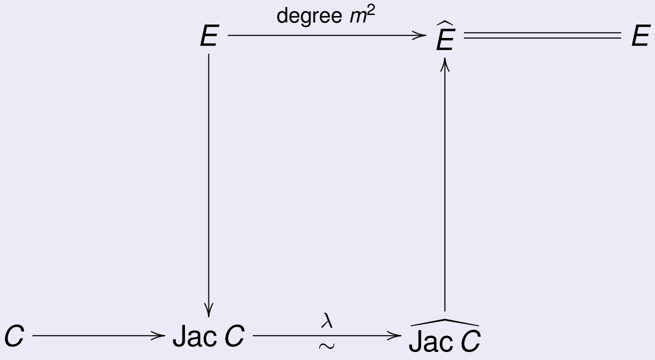
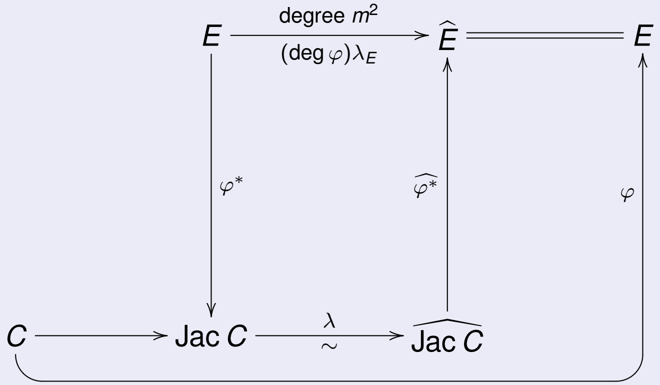


The big diagram when $n > 1$.



So $\deg \varphi = m$. We need bounds on the length of the shortest vector in a Hermitian lattice with a given Gram determinant.

An example.

Possible real Weil polynomial for $q = 4$, $g = 7$, $N = 22$:
$h = h_1 h_2$ with $h_1 = (x+3)^3$ and $h_2 = x(x+2)^2(x+4)$.

Let E have real Weil polynomial $x + 3$.
E has complex multiplication by $\mathcal{O} = \mathbb{Z}[(1+\sqrt{-7})/2]$.

We deduce. . .

- a polarization of degree 9 on E^3; and therefore
- a Hermitian form H on $\mathcal{O}^3$ with $\det \operatorname{Gram} H = 3$.

If H has vector of squared-length 2, we get double cover C (with 22 points) $\to E$ (with 8 points), contradiction.

Note: Vector of squared-length 3 doesn't help us.

From a Hermitian form to a positive quadratic form.

View $\mathcal{O}$ as $\mathbb{Z} \oplus \mathbb{Z}$. Then H gives us
an integer-valued positive definite quadratic form P on $\mathbb{Z}^6$.

(Note: The associated bilinear form is the real part of H, which is *half*-integer valued.)

- $\det \mathrm{Gram}\, P = N_{\mathcal{O}/\mathbb{Z}}(\det \mathrm{Gram}\, H) \cdot |\operatorname{disc} \mathcal{O}/4|^3 = 3087/64$.
- Let $M_1, \ldots, M_6$ be successive minima of P. Then

$$M_1 \cdots M_6 \leq (64/3)(3087/64) = 1029.$$

- If no vectors of squared-length 1 or 2, then

$$M_1 = M_2 = M_3 = M_4 = M_5 = 3 \quad \text{and} \quad 3 \leq M_6 \leq 4.$$

Back to the Hermitian form.

The first 5 minima generate a $\mathbb{Q}$-vector space of dimension 5. So they must generate a $\mathbb{Q}(\sqrt{-7})$-vector space of dimension 3.

Let $v_1, v_2, v_3 \in \mathcal{O}^3$ be $\mathbb{Q}(\sqrt{-7})$-independent vectors of squared-length 3.

Let Λ be $\mathcal{O}$-sublattice of $\mathcal{O}^3$ generated by v_1, v_2, v_3.

$$\operatorname{Gram} H|_\Lambda = \begin{bmatrix} 3 & a & b \\ \overline{a} & 3 & c \\ \overline{b} & \overline{c} & 3 \end{bmatrix}$$

$$\det \operatorname{Gram} H|_\Lambda = (\det \operatorname{Gram} H) \cdot N_{\mathcal{O}/\mathbb{Z}}([\mathcal{O}^3/\Lambda]).$$

positive definite $\Longrightarrow$ a, b, c have norm less than 9.

A small finite problem.

Algorithm to find bad forms:

- Enumerate all possible (a, b, c).
- For each triple: Does associated matrix have determinant $3N(\mathfrak{A})$ for an ideal $\mathfrak{A}$ of $\mathcal{O}$?
- If so, find all superlattices on which form has determinant 3.
- Compute shortest vector v in each superlattice.
- If v has squared-length 3, we have a bad example.

We found no bad examples.

Every polarization of degree 9 on E^3 can be pulled back to a polarization of degree 1 or 4 on E.

Remark.

This procedure does not scale well to higher dimensions.

When $\det \mathrm{Gram}\, H$ is a norm from $\mathcal{O}$, there is a better procedure.

- Based on Schiemann's calculation of all unimodular forms on $\mathcal{O}^n$ for small n and small $\mathcal{O}$.
- When $\det \mathrm{Gram}\, H$ is a norm, there is a superlattice on which H is unimodular.

Sample optimal bounds.

For the quadratic order $\mathcal{O}$ of discriminant -7:

dim\det	1	2	3	4	5	6	7	8	9	10	11	12
2	1	2	2	2	3	2	3	4	4	3	3	4
3	2	2	2	2	2	2	3	4	3	2	3	3
4	2	2		2			3	2	3		3	
5	2	2		2			3	2	3		3	

Sharp upper bounds on the squared-lengths of short vectors for Hermitian forms over $\mathcal{O}$ of a given dimension and determinant.

Computer calculations.

Pari/GP code

- Given q, g, N, enumerates all polynomials h with
  - leading terms $x^g + (N - q - 1)x^{g-1} + \cdots$, and
  - all complex roots in $[-2\sqrt{q}, 2\sqrt{q}]$.
  - Uses ideas of McKee and Smyth (ANTS 2004).
- Eliminates those that are not Weil polynomials.
- Computes all possible splittings $h = h_1 h_2$.
  - Computes modifed reduced resultant of each splitting.
- Applies Serre's "reduced resultant 1" criterion.
- Applies "reduced resultant 2" method.
- Applies generalized elliptic factor method.
- If either method gives a cover $C \to D$, checks some conditions to see whether such a cover is possible.

New results.

Upper and lower bounds on $N_q(g)$, as of November 2006.

$g \setminus q$	2	4	8	16	32	64	128
1	5	9	14	25	44	81	150
2	6	10	18	33	53	97	172
3	7	14	24	38	64	113	192
4	8	15	25	45	71 – 74	129	215
5	9	17	29 – 30	49 – 53	83 – 85	132 – 145	227 – 234
6	10	20	33 – 35	65	86 – 96	161	243 – 258
7	10	21 – 22	34 – 38	63 – 69	98 – 107	177	262 – 283
8	11	21 – 24	35 – 42	62 – 75	97 – 118	169 – 193	276 – 302
9	12	26	45	72 – 81	108 – 128	209	288 – 322
10	13	27	42 – 49	81 – 87	113 – 139	225	296 – 345
11	14	26 – 29	48 – 53	80 – 91	120 – 150	201 – 236	294 – 366
12	14 – 15	29 – 31	49 – 57	88 – 97	129 – 161	257	321 – 388
13	15	33	56 – 61	97 – 102	129 – 172	255 – 268	– 408
14	15 – 16	32 – 35	65	97 – 107	146 – 183	241 – 284	353 – 437
15	17	35 – 37	57 – 67	98 – 113	158 – 194	258 – 300	386 – 455

New results.

Current upper bounds, lower bounds from November 2006.

$g \setminus q$	2	4	8	16	32	64	128
1	5	9	14	25	44	81	150
2	6	10	18	33	53	97	172
3	7	14	24	38	64	113	192
4	8	15	25	45	71 – 72	129	215
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Tempting partial results.

Genus-12 curves over $\mathbb{F}_2$ with 15 points:

Code examined 22 possible polynomials.

- All satisfied Honda-Tate conditions.
- 10 failed "reduced resultant 1" test.
- 7 failed "reduced resultant 2" test.
- None failed "generalized elliptic factor" test.
- 3 were eliminated by *ad hoc* methods.

Only two possible real Weil polynomials:

- $(x+1)^2(x+2)^2(x^2-2)(x^2+2x-2)^3$
- $(x^2+x-3)(x^3+3x^2-3)(x^3+4x^2+3x-1)(x^4+4x^3+2x^2-5x-3)$

First has degree-4 map to elliptic curve with 4 points.
Second has $\mathbb{F}_{2^7}$-rational degree-4 map to elliptic curve over $\mathbb{F}_2$ with 2 points.